BV-STRUCTURE OF THE COHOMOLOGY OF NILPOTENT SUBALGEBRAS
AND THE GEOMETRY OF ($\mathcal{W}$-) STRINGS

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ABSTRACT

Given a simple, simply laced, complex Lie algebra $\mathfrak{g}$ corresponding to the Lie group G, let $\mathfrak{n}_+$ be the subalgebra generated by the positive roots. In this paper we construct a BV-algebra BV[$\mathfrak{g}$] whose underlying graded commutative algebra is given by the cohomology, with respect to $\mathfrak{n}_+$, of the algebra of regular functions on G with values in $\bigwedge(\mathfrak{n}_+\backslash\mathfrak{g})$. We conjecture that BV[$\mathfrak{g}$] describes the algebra of *all* physical (i.e., BRST invariant) operators of the noncritical $\mathcal{W}$[$\mathfrak{g}$] string. The conjecture is verified in the two explicitly known cases, $\mathfrak{g} = \mathfrak{sl}_2$ (the Virasoro string) and $\mathfrak{g} = \mathfrak{sl}_3$ (the $\mathcal{W}_3$ string).

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1. Introduction

In the past few years there has been a considerable effort to understand algebraic and geometric structures underlying $\mathcal{W}$-gravity in two dimensions. The initial observation [1,2] was that a subsector of the algebra of physical (i.e., BRST invariant) operators $\mathfrak{H}[\mathcal{W}_2]$ of the $\mathcal{W}_2$ (Virasoro) string is modeled by polynomial polyvectors on the complex plane. Subsequent work [3,4,5] revealed that the full algebra of physical operators has the structure of a Batalin-Vilkovisky (BV-) algebra, with the BV-algebra of polyvectors arising as a quotient determined by the natural action of $\mathfrak{H}[\mathcal{W}_2]$ on the ground ring.

In a recent paper [6] we found a similar, albeit significantly more complicated, description of the operator algebra of the $\mathcal{W}_3$ string. Crucial to our construction was the observation that the proper geometric framework for studying the $\mathcal{W}_n$ string ($n = 2, 3$) is the base affine space [7] of $SL(n)$. The operator algebra, $\mathfrak{H}[\mathcal{W}_n]$, can then be parametrized as a direct sum of (twisted) polyderivations of the ground ring, and contains the BV-algebra of polyvectors on the base affine space as its quotient. However, a simple characterization of the full BV-algebra of the $\mathcal{W}_n$ string in terms of geometric objects associated with $\mathfrak{sl}_n$ and its base affine space was not known.

In this letter we will provide the desired characterization. We will show that with any (simply-laced) Lie algebra $\mathfrak{g}$ one may associate a BV-algebra, BV[$\mathfrak{g}$], whose underlying graded commutative algebra is given by the cohomology, $H(\mathfrak{n}_+, \mathcal{E}(G) \otimes \bigwedge(\mathfrak{n}_+ \backslash \mathfrak{g}))$, with respect to the maximal nilpotent subalgebra $\mathfrak{n}_+ \subset \mathfrak{g}$, where $\mathcal{E}(G)$ is the space of regular functions on the complex Lie group G of $\mathfrak{g}$. Since the polyvectors on the base affine space of G are characterized as the $\mathfrak{n}_+$-invariant elements in $\mathcal{E}(G) \otimes \bigwedge(\mathfrak{n}_+ \backslash \mathfrak{g})$, they are incorporated into this picture as the zero-th order cohomology. For the $\mathcal{W}$-strings whose spectra are explicitly known, the $\mathcal{W}_2$ string [8,9] and the $\mathcal{W}_3$ string [6], we verify that the higher order cohomologies account correctly for the remaining sectors of the spectrum. This gives a simple geometric picture for the operator algebra of rather complicated models without recourse to generalized polyvectors.

This result is also of interest since it relates the semi-infinite cohomology of an infinite-dimensional $\mathcal{W}$-algebra with coefficients in infinite-dimensional modules (Fock modules), to the Lie algebra cohomology of a finite-dimensional Lie algebra $\mathfrak{n}_+$ with coefficients in $L(\Lambda) \otimes \bigwedge(\mathfrak{n}_+ \backslash \mathfrak{g})$, where $L(\Lambda)$, $\Lambda \in P_+$, is a finite-dimensional irreducible module of $\mathfrak{g}$. (The irreducible modules arise upon decomposition of $\mathcal{E}(G)$ with respect to $\mathfrak{g}$.) To determine the latter cohomology we are led to a classical problem in Lie algebra cohomology; namely, the computation of $H(\mathfrak{n}_+, L(\Lambda))$ [10]. Here we will argue that for a weight $\Lambda \in P_+$ sufficiently deep inside the fundamental Weyl chamber,

$$H(\mathfrak{n}_+, L(\Lambda) \otimes \bigwedge(\mathfrak{n}_+ \backslash \mathfrak{g})) \;\cong\; H(\mathfrak{n}_+, L(\Lambda)) \otimes \bigwedge(\mathfrak{n}_+ \backslash \mathfrak{g}) \,. \tag{1.1}$$

The result in this paper is consistent with the important role played by $\mathfrak{n}_+$-cohomology in $\mathcal{W}$-gravity noticed earlier [11] for the so-called "generic" regime of central charge. The explanation of these observations for $\mathcal{W}$-gravities is expected to be found in the context of Hamiltonian reduction, or G/G models, but their precise origin is still unknown.

Throughout this letter we use the conventions and notation of [6].

2. BV-structure of $H(\mathfrak{n}_+, \mathcal{E}(G) \otimes \bigwedge \mathfrak{b}_-)$

Let $\mathfrak{g}$ be a complex, simple, simply-laced, finite dimensional Lie algebra. We fix a Cartan decomposition $\mathfrak{g} \cong \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \cong \mathfrak{b}_- \oplus \mathfrak{n}_+$ and denote the corresponding Chevalley generators of $\mathfrak{g}$ by $\{e_{-\alpha}, h_i, e_\alpha\}$, $i = 1, \ldots, \ell \equiv \text{rank } \mathfrak{g}$, $\alpha \in \Delta_+$. The collective indices of $\mathfrak{b}_-$ and $\mathfrak{g}$ will be denoted by $a = (-\alpha, i)$ and

$A = (-\alpha, i, \alpha)$, respectively, and the structure constants of $\mathfrak{g}$ in the chosen basis by $f_{AB}{}^C$. We use the summation convention in which the repeated indices are summed over their ranges.

The space, $\mathcal{E}(G)$, of regular functions on the complex Lie group G of $\mathfrak{g}$ carries the left and right regular representations of $\mathfrak{g}$ corresponding to the left and right invariant vector fields on G. Denote the operators representing the action of a generator e_A by Π_A^R and Π_A^L, respectively. The Peter-Weyl theorem asserts that as a $\mathfrak{g}_L \oplus \mathfrak{g}_R$ module

$$\mathcal{E}(G) \cong \bigoplus_{\Lambda \in P_+} L(\Lambda^*) \otimes L(\Lambda), \tag{2.1}$$

where $L(\Lambda)$ and $L(\Lambda^*)$ are the irreducible finite dimensional $\mathfrak{g}$-modules with highest (dominant integral) weights Λ and Λ^*, respectively, and $\Lambda^* = -w_0 \Lambda$.

Following [7], we define the base affine space of G as the quotient $A = N_+ \backslash G$, where N_+ is the subgroup generated by $\mathfrak{n}_+$. The aim of this section is to study geometric objects that are associated with A. The basic example is the space of polyvectors, $\mathfrak{P}(A)$ defined as regular sections of the homogenous vector bundle $G \times_{N_+} \bigwedge \mathfrak{b}_-$. Here we consider $\bigwedge \mathfrak{b}_-$ as an $\mathfrak{n}_+$-module through the identification $\bigwedge \mathfrak{b}_- \cong \bigwedge(\mathfrak{n}_+ \backslash \mathfrak{g})$. Clearly there is a natural grading $\mathfrak{P}(A) = \bigoplus_n \mathfrak{P}^n(A)$ induced from the decomposition

$$\bigwedge \mathfrak{b}_- \cong \bigoplus_{n=0}^D \bigwedge{}^n \mathfrak{b}_-, \qquad D = |\Delta_+| + \ell.$$

At this point it convenient to introduce a set of ghost oscillators $\{b^a, c_a\}$, corresponding to $\mathfrak{b}_-$, with nontrivial anti-commutators $[b^a, c_b] = \delta^a{}_b$ and associated ghost Fock space F^{bc} with vacuum $|bc\rangle$ satisfying $b^a |bc\rangle = 0$. We identify $\bigwedge \mathfrak{b}_-$ with F^{bc}, with the $\mathfrak{n}_+$ action given by $\Pi_\alpha^{bc} = f_{\alpha b}{}^c c_c b^b$. In particular this identification induces a graded commutative product on F^{bc}. Moreover, F^{bc} is also an $\mathfrak{h}$-module with the $\mathfrak{h}$ generators $\Pi_i^{bc} = f_{ib}{}^c c_c b^b$.

Let $\mathcal{E}(G) \otimes \bigwedge \mathfrak{b}_-$ denote the space of regular functions on G with values in $\bigwedge \mathfrak{b}_-$. It has a natural structure of a graded, graded commutative algebra and carries commuting actions of $\mathfrak{g} \oplus \mathfrak{h}$ and $\mathfrak{n}_+$, defined by the operators Π_A^R and $\Pi_i^L + \Pi_i^{bc}$, and $\Pi_\alpha^L + \Pi_\alpha^{bc}$, respectively. Both $\mathfrak{g} \oplus \mathfrak{h}$ and $\mathfrak{n}_+$ act by derivations of the algebra product. In terms of $\mathcal{E}(G) \otimes \bigwedge \mathfrak{b}_-$ the polyvectors $\mathfrak{P}(A)$ are simply given by the $\mathfrak{n}_+$-invariant elements, i.e.,

$$\mathfrak{P}^n(A) \cong (\mathcal{E}(G) \otimes \bigwedge{}^n \mathfrak{b}_-)^{\mathfrak{n}_+}. \tag{2.2}$$

We note that polyvectors of order 0 are simply identified with the regular functions, $\mathcal{E}(A)$, on A. Using (2.1), the decomposition of $\mathcal{E}(A)$ with respect to $\mathfrak{h} \oplus \mathfrak{g}$ yields

$$\mathcal{E}(A) \cong \bigoplus_{\Lambda \in P_+} \mathbb{C}_{\Lambda^*} \otimes L(\Lambda), \tag{2.3}$$

i.e., $\mathcal{E}(A)$ is a model space of $\mathfrak{g}$. The computation of higher order polyvectors is more involved due to the typically reducible, but indecomposable, action of $\mathfrak{n}_+$ on $\bigwedge \mathfrak{b}_-$ (see [7,6]).

The natural framework for determining invariants of group actions is Lie algebra cohomology. In this more general context, the polyvectors are obtained as the zero-th order cohomology of $\mathfrak{n}_+$ with coefficients in $\mathcal{E}(G) \otimes \bigwedge \mathfrak{b}_-$. One is naturally led to consider the algebra, BV[$\mathfrak{g}$], defined by the full cohomology,

$$\mathrm{BV}[\mathfrak{g}] \equiv H(\mathfrak{n}_+, \mathcal{E}(G) \otimes \bigwedge \mathfrak{b}_-). \tag{2.4}$$

Let us now examine BV[$\mathfrak{g}$] more closely, using this as an opportunity to introduce further notation and to derive some elementary results. We introduce a set of ghost oscillators $\{\sigma^\alpha, \omega_\alpha\}$, corresponding to $\mathfrak{n}_+$, with

nontrivial anti-commutators $[\sigma^\alpha, \omega_\beta] = \delta^\alpha_\beta$ and associated ghost Fock space $F^{\sigma\omega}$ with vacuum $|\sigma\omega\rangle$ satisfying $\omega_\alpha|\sigma\omega\rangle = 0$. The $\mathfrak{n}_+$ action on $F^{\sigma\omega}$ is given by $\Pi^{\sigma\omega}_\alpha = -f_{\alpha\beta}{}^\gamma \sigma^\beta \omega_\gamma$, and the $\mathfrak{h}$ action by $\Pi^{\sigma\omega}_i = -f_{i\alpha}{}^\alpha \sigma^\alpha \omega_\alpha$. Again, there is a natural graded commutative product on $F^{\sigma\omega}$.

The cohomology $H(\mathfrak{n}_+, \mathcal{E}(G) \otimes \bigwedge \mathfrak{b}_-)$ may now be computed as the cohomology of the differential

$$d = \sigma^\alpha \left(\Pi^L_\alpha + \Pi^{bc}_\alpha + \tfrac{1}{2}\Pi^{\sigma\omega}_\alpha \right) \tag{2.5}$$

acting on the complex $\mathcal{C}(G) \equiv \mathcal{E}(G) \otimes F^{bc} \otimes F^{\sigma\omega}$, i.e. $H_d(\mathcal{C}(G))$. The complex is bi-graded by the bc-ghost and the $\sigma\omega$-ghost numbers ($\mathrm{gh}(c_a) = -\mathrm{gh}(b^a) = (0,1)$, $\mathrm{gh}(\sigma^\alpha) = -\mathrm{gh}(\omega_\alpha) = (1,0)$), with d of degree $(1,0)$. Clearly, this bi-degree passes to the cohomology. We will write $H^n(\mathfrak{n}_+, \mathcal{E}(G) \otimes \bigwedge \mathfrak{b}_-)$ for the cohomology in *total* ghost number n.

Combining the above discussions, the complex $\mathcal{C}(G)$ is a $\mathfrak{g} \oplus \mathfrak{h}$-module under Π^R_A and

$$\Pi_i \equiv \Pi^L_i + \Pi^{bc}_i + \Pi^{\sigma\omega}_i . \tag{2.6}$$

Note that with respect to this $\mathfrak{h}$-action, the weights of $b^{-\alpha}$ and ω_α are α, whilst those of σ^α and $c_{-\alpha}$ are $-\alpha$. Since d commutes both with the action of $\mathfrak{g}$ and $\mathfrak{h}$, we have a direct sum decomposition

$$\mathrm{BV}[\mathfrak{g}] = \bigoplus_{\Lambda \in P_+} H(\mathfrak{n}_+, L(\Lambda^*) \otimes \bigwedge \mathfrak{b}_-) \otimes L(\Lambda) . \tag{2.7}$$

As an $\mathfrak{h}$-module,

$$H(\mathfrak{n}_+, L(\Lambda) \otimes \bigwedge \mathfrak{b}_-) \cong \bigoplus_{\lambda \in P(\mathcal{C}(\Lambda))} H(\mathfrak{n}_+, L(\Lambda) \otimes \bigwedge \mathfrak{b}_-)_\lambda , \tag{2.8}$$

where, obviously, $\mathcal{C}(\Lambda) = L(\Lambda) \otimes F^{bc} \otimes F^{\sigma\omega}$, and $P(V)$ denotes the set of weights of an $\mathfrak{h}$-module V.

The decomposition (2.7) reduces the problem of computing $\mathrm{BV}[\mathfrak{g}]$ to that of computing cohomology of finite-dimensional modules. We postpone a more detailed discussion till Section 3 and first concentrate on global properties of $\mathrm{BV}[\mathfrak{g}]$.

Lemma 2.1. $\mathrm{BV}[\mathfrak{g}]$ *is a graded, graded commutative algebra with the product "$\cdot$" induced from the product on the underlying complex $\mathcal{C}(G)$.*

Proof: Let $|0\rangle = |bc\rangle \otimes |\sigma\omega\rangle$. Then $\Phi \in \mathcal{C}(G)$ is of the form

$$\Phi = \Phi^{a_1 \ldots a_m}{}_{\alpha_1 \ldots \alpha_n} \sigma^{\alpha_1} \ldots \sigma^{\alpha_n} c_{a_1} \ldots c_{a_m} |0\rangle , \qquad \Phi^{a_1 \ldots a_m}{}_{\alpha_1 \ldots \alpha_n} \in \mathcal{E}(G) . \tag{2.9}$$

The product of two such element is thus given by

$$\Phi \cdot \Psi = (-1)^{np} \Phi^{a_1 \ldots a_m}{}_{\alpha_1 \ldots \alpha_n} \Psi^{b_1 \ldots b_p}{}_{\beta_1 \ldots \beta_q} \sigma^{\alpha_1} \ldots \sigma^{\alpha_n} \sigma^{\beta_1} \ldots \sigma^{\beta_q} c_{a_1} \ldots c_{a_m} c_{b_1} \ldots c_{b_p} |0\rangle . \tag{2.10}$$

We verify that

$$d(\Phi \cdot \Psi) = d\Phi \cdot \Psi + (-1)^{m+n} \Phi \cdot d\Psi , \tag{2.11}$$

from which it follows immediately that the product passes to the cohomology. Obviously, it is graded commutative according to the total ghost number. $\square$

The algebra $\mathrm{BV}[\mathfrak{g}]$ contains the algebra of polyvectors $\mathfrak{P}(A)$ as a subalgebra, namely

$$\mathfrak{P}^m(A) \cong \mathrm{BV}^{(0,m)}[\mathfrak{g}] . \tag{2.12}$$

Let $\imath : \mathfrak{P}(A) \longrightarrow \mathrm{BV}[\mathfrak{g}]$ be the corresponding embedding. Then we have $\imath(\mathfrak{P}(A)) = \bigcap_\alpha \mathrm{Ker}\, \omega_\alpha$, and compatibility with the cohomology requires that $\imath(\mathfrak{P}(A))$ be $\mathfrak{n}_+$-invariant.

The space $\mathfrak{P}(A)$ also carries an additional BV-algebra structure, see e.g. [6]. We will now prove that there are natural extensions of the BV-structure from $\mathfrak{P}(A)$ to $\mathrm{BV}[\mathfrak{g}]$.

Recall that a BV-algebra $(\mathrm{BV}, \cdot, \Delta)$ consists of a graded, graded commutative algebra $(\mathrm{BV}, \cdot)$ with an operator Δ, called the BV-operator, which is a graded second order derivation of degree -1 on BV satisfying $\Delta^2 = 0$. We refer the reader to [3,4,5,6] for further details.

We will introduce the BV-algebra structure on $\mathrm{BV}[\mathfrak{g}]$ in two steps. First, we will show that there is a natural extension of the BV-operator from $\mathfrak{P}(A)$ to $\mathrm{BV}[\mathfrak{g}]$ that preserves the space of polyvectors, but has nontrivial cohomology. Secondly, we will construct a deformation of the "naive" BV-operator such that the cohomology becomes trivial. It is the latter BV-structure that turns out to be relevant for $\mathcal{W}[\mathfrak{g}]$ strings, as will be discussed in Section 4.

Theorem 2.2. *Consider the operator*

$$\Delta_0 \;=\; -b^a(\Pi_a^L + \Pi_a^{\sigma\omega}) + \tfrac{1}{2}f_{ab}{}^c b^a b^b c_c \,, \tag{2.13}$$

where $\Pi_a^{\sigma\omega} = -f_{a\alpha}^\beta b^a \sigma^\alpha \omega_\beta$. Then

  i. $[d, \Delta_0] = 0$,
  ii. Δ_0 *is a BV-operator on* $\mathrm{BV}[\mathfrak{g}]$,
  iii. $\Delta_0 \imath(\mathfrak{P}(A)) \subset \imath(\mathfrak{P}(A))$.

Proof: Note that $-\Delta_0$ is the differential of $\mathfrak{b}_-$-cohomology with coefficients in $\mathcal{E}(G) \otimes F^{\sigma\omega}$. Thus $\Delta_0^2 = 0$, a fact easily verified by an explicit algebra. Moreover, if we combine the $\mathfrak{b}_-$ and $\mathfrak{n}_+$ ghosts as $c^A = \{c_a, \omega_\alpha\}$ and $b^A = \{b^a, \sigma^\alpha\}$ then

$$\delta \;=\; d - \Delta_0 = b^A \Pi_A^L - \tfrac{1}{2}f_{AB}^C b^A b^B c_C \,, \tag{2.14}$$

is the differential of a twisted cohomology [12] of $\mathfrak{g}$ with coefficients in $\mathcal{E}(G)$. Thus $\delta^2 = 0$ and the assertion (i) follows.

The second order derivation property of Δ_0 is shown as follows: The first term in Δ_0 is a product of first order derivations $\Pi_a^L + \Pi_a^{\sigma\omega}$ on $\mathcal{E}(G) \otimes F^{\sigma\omega}$ and b^a on F^{bc}. The product of such first order derivations acting on the tensor product of spaces is well-known to be a second order derivation. The second term, with the nontrivial action on F^{bc} only, upon normal ordering of the bc-ghosts becomes a sum of terms with one or a product of two b^a's, which are first and second order derivations, respectively. Since we have already shown that $\Delta_0^2 = 0$, this proves (ii).

Part (iii) follows from the observation that on $\imath(\mathfrak{P}(A))$ the only term involving the $\sigma\omega$-ghosts vanishes. Then Δ_0 reduces to the BV-operator on $\mathfrak{P}(A)$ constructed in [6]. $\square$

We will now seek a deformation of Δ_0 of the form $\Delta = \Delta_0 + \Delta_1$, such that Δ is a BV-operator on $\mathrm{BV}[\mathfrak{g}]$. In particular this requires that $[d, \Delta_1] = 0$, which implies that Δ_1 must be a nontrivial element in the "operator cohomology" of d on $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(bc) \otimes \mathcal{U}(\sigma\omega)$. Here $\mathcal{U}(\cdot)$ denotes the enveloping algebra and d acts by the commutator. Since computing this cohomology is difficult, we make the further simplification that Δ_1 is an element of $\mathcal{U}(bc) \otimes \mathcal{U}(\sigma\omega)$. This assumption is motivated by the naive expectation that the only second order derivation that has degree -1 and acts nontrivially on the $\mathcal{E}(G)$ component in $\mathrm{BV}[\mathfrak{g}]$ must be of the form $b^a \Pi_a^L$, and thus is already accounted for in Δ_0. Then we have

Theorem 2.3. *Let $\mathfrak{g}$ be a simple, simply laced Lie algebra and $\epsilon : Q \times Q \longrightarrow \{\pm 1\}$ its asymmetry function with respect to the chosen Chevalley basis.*[1] *Define*

$$\Delta' = \sum_{\alpha \in \Delta_+} \sum_{i=1}^{\ell} (\alpha, \alpha_i) \, \sigma^{\alpha} b^{-\alpha} b^i - \sum_{\alpha, \beta \in \Delta_+} \epsilon(\alpha, \beta) \sigma^{\alpha+\beta} b^{-\alpha} b^{-\beta} \,. \tag{2.15}$$

Then $\Delta_t = \Delta_0 + t\Delta'$ is a one parameter family of BV-operators on $\mathrm{BV}[\mathfrak{g}]$.

Proof: The required properties are proved by explicit algebra. The details will be presented in the revised version of [6]. $\square$

Remarks:

1. We have verified that for $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{sl}_3$ the generator Δ' of the one parameter family of deformations of Δ_0 is uniquely determined by requiring that it be a nontrivial element of the $\mathfrak{n}_+$-cohomology at ghost number -1. It is an interesting open problem to determine whether this also true for an arbitrary $\mathfrak{g}$.
2. The assumption that $\mathfrak{g}$ is simply laced is merely technical, and it should be straightforward to obtain a generalization of (2.15) to the non-simply laced case.

A simple scaling argument shows that in fact all BV-algebra structures on $\mathrm{BV}[\mathfrak{g}]$ for $t \neq 0$ are equivalent. Indeed, if we let $\sigma^{\alpha} \rightarrow \lambda\sigma^{\alpha}$, $\omega_{\alpha} \rightarrow \lambda^{-1}\omega_{\alpha}$ then $d \rightarrow \lambda d$, $\Delta_0 \rightarrow \Delta_0$ while $\Delta' \rightarrow \lambda\Delta'$. Since the cohomology classes must scale homogenously with respect to this transformation, we may use it to set $t = 1$, and denote $\Delta = \Delta_{t=1}$.

Finally, we have shown in [6] that the cohomology of Δ_0 on $\mathfrak{P}(A)$ is nontrivial and spanned by the volume element $\prod_a c_a |bc\rangle$. We now have

Conjecture 2.4. $\mathrm{BV}[\mathfrak{g}]$ *is acyclic with respect to Δ.*

This conjecture holds in the case of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$, where $\mathrm{BV}[\mathfrak{g}]$ can be computed explicitly as we now show.

3. Computation of $H(\mathfrak{n}_+, L(\Lambda) \otimes \bigwedge \mathfrak{b}_-)$

In this section we will compute the cohomology $H(\mathfrak{n}_+, L(\Lambda) \otimes \bigwedge \mathfrak{b}_-)$ for $\Lambda \in P_+$ which, by (2.7), implies the result for $H(\mathfrak{n}_+, \mathcal{E}(G) \otimes \bigwedge \mathfrak{b}_-)$.

If Λ is sufficiently deep inside the fundamental Weyl chamber (henceforth we refer to such Λ as "in the bulk") the cohomology is easily computed. The result is given by the following

Theorem 3.1. *Let $\Lambda \in P_+$*

  i. *The cohomology $H^n(\mathfrak{n}_+, L(\Lambda) \otimes \bigwedge \mathfrak{b}_-)_{\Lambda'}$ is nontrivial only if there exists a $w \in W$ and $\lambda \in P(\bigwedge^k \mathfrak{b}_-)$ such that $\Lambda' = w(\Lambda + \rho) - \rho + \lambda$ and $n = \ell(w) + k$.*
  ii. *For $\Lambda \in P_+$ in the bulk, i.e. $(\Lambda, \alpha_i) \geq N(\mathfrak{g})$ for some $N(\mathfrak{g}) \in \mathbb{N}$ sufficiently large (in particular $N(\mathfrak{sl}_n) = n - 1$), we have*

$$H(\mathfrak{n}_+, L(\Lambda) \otimes \textstyle\bigwedge \mathfrak{b}_-) \cong H(\mathfrak{n}_+, L(\Lambda)) \otimes \textstyle\bigwedge \mathfrak{b}_- \cong \left(\bigoplus_{w \in W} \mathbb{C}_{w(\Lambda+\rho)-\rho} \right) \otimes \textstyle\bigwedge \mathfrak{b}_- \,. \tag{3.1}$$

Proof: Consider the gradation of the complex $\mathcal{C} = \oplus_{k \in \mathbb{Z}} \mathcal{C}_k$ given by (ρ, λ^{bc}), where λ^{bc} is the weight corresponding to Π_i^{bc}. With respect to this gradation the differential (2.5) decomposes as

$$d = \sum_{k \geq 0} d_k \,,$$

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[1] Recall that $\epsilon(\alpha, \beta) = f_{\alpha\beta}{}^{\gamma}$, $\alpha, \beta, \gamma \in \Delta$, see e.g. [13].

with

$$d_0 \;=\; \sigma^\alpha \left(\Pi^L_\alpha + \tfrac{1}{2} \Pi^{\sigma\omega}_\alpha \right), \qquad d_k \;=\; \sum_{(\rho,\alpha)=k} \sigma^\alpha \Pi^{bc}_\alpha, \qquad k \geq 1.$$

In particular, $d_k \equiv 0$ for $k \geq (\rho,\theta) + 1 = \mathrm{h}^\vee$, where $\mathrm{h}^\vee$ is the dual Coxeter number of $\mathfrak{g}$ (for $\mathfrak{g} = \mathfrak{sl}_n$ we have $\mathrm{h}^\vee = n$). The spectral sequence (E_k, δ_k) corresponding to this gradation converges since the complex is finite dimensional (see e.g. [14] for an elementary exposition). The first term in the spectral sequence is given by

$$E_1 \;=\; H_{d_0}(\mathcal{C}) \;\cong\; H(\mathfrak{n}_+, L(\Lambda)) \otimes \textstyle\bigwedge \mathfrak{b}_- \,,$$

where $H(\mathfrak{n}_+, L(\Lambda)) \cong \oplus_{w \in W} \mathbb{C}_{w(\Lambda+\rho)-\rho}$ [10]. This proves the first part of the theorem. To prove the second part we will now determine the condition for the spectral sequence to collapse at the first term.

The differential δ_1 on E_1 is simply given by d_1, so we find that a sufficient condition for δ_1 to act trivially is that there exist no $w, w' \in W$, $\ell(w') = \ell(w) + 1$, $\lambda, \lambda' \in P(\bigwedge \mathfrak{b}_-)$ such that

$$w(\Lambda + \rho) - w'(\Lambda + \rho) \;=\; \lambda' - \lambda \;=\; \alpha_i \,,$$

for some i. Similarly, we find that a sufficient condition for δ_k to act trivially on E_k is that there exist no $w, w' \in W$, $\ell(w') = \ell(w) + 1$, $\lambda, \lambda' \in P(\bigwedge \mathfrak{b}_-)$ such that

$$w * \Lambda - w' * \Lambda \;=\; \lambda' - \lambda \;\in\; \{\beta \in Q_+ \,|\, (\rho,\beta) \leq k\}\,.$$

So, since $\delta_k \equiv 0$ for $k \geq \mathrm{h}^\vee$, we find that a sufficient condition for $E_\infty \cong \ldots \cong E_2 \cong E_1$ is that there exist no $w, w' \in W$, $\ell(w') = \ell(w) + 1$, $\lambda, \lambda' \in P(\bigwedge \mathfrak{b}_-)$ such that

$$w * \Lambda - w' * \Lambda \;=\; \lambda' - \lambda \;\in\; \{\beta \in Q_+ \,|\, (\rho,\beta) \leq \mathrm{h}^\vee - 1\}\,.$$

This condition is met when Λ is sufficiently deep inside the fundamental Weyl chamber, i.e. $(\Lambda, \alpha_i) \geq N(\mathfrak{g})$ for some $N(\mathfrak{g}) \in \mathbb{N}$ sufficiently large. $\square$

In principle the cohomology for Λ away from the bulk can be computed by explicitly going through the spectral sequence discussed above. For $\mathfrak{sl}_2$ this is particularly easy since only the singlet weight $\Lambda = 0$ is not in the bulk. In this case one easily verifies that the states $c_{-\alpha}|bc\rangle \otimes |\sigma\omega\rangle$ and $c_1 |bc\rangle \otimes \sigma^\alpha |\sigma\omega\rangle$ in E_1 are eliminated in E_2. In other words, for $\mathfrak{sl}_2$ one finds that $H(\mathfrak{n}_+, L(\Lambda) \otimes \bigwedge \mathfrak{b}_-)$ for $\Lambda = 0$ contains six states (organized in three doublets at ghost numbers $0, 1$ and 2 and $\mathfrak{h}$-weights $0, -\alpha, -2\alpha$, respectively, where a doublet at ghost number n is a pair of states of the same weight at ghost numbers n and $n + 1$), as opposed to eight states (Theorem 3.1) for $\Lambda \neq 0$.

For algebras other than $\mathfrak{sl}_2$, going through the above spectral sequence becomes rather cumbersome. Instead we will present an independent calculation, based on free field techniques, for the other case of special interest – namely, $\mathfrak{sl}_3$. Here we will determine the cohomology $H(\mathfrak{n}_+, L(\Lambda) \otimes \bigwedge \mathfrak{b}_-)$, for arbitrary $\Lambda \in P_+$, through a spectral sequence associated with the resolution $(\mathcal{F}, \delta)$ of the irreducible module $L(\Lambda)$ in terms of free field Fock spaces which are co-free over $\mathfrak{n}_+$, i.e. isomorphic to contragradient Verma modules. The reader should consult [12,15] for a review of such techniques, but for completeness we will recall the little of this theory which is required here.

Introduce the free oscillators $\beta_\alpha, \gamma^\alpha$, $\alpha \in \Delta_+$, with nontrivial commutators $[\gamma^\alpha, \beta_\beta] = \delta^\alpha{}_\beta$ and associated Fock space $F^{\beta\gamma}_\Lambda$ with vacuum $|\Lambda\rangle$, $\Lambda \in P$, satisfying $\beta_\alpha |\Lambda\rangle = 0$. Then, $F^{\beta\gamma}_\Lambda$ can be given the structure of a $\mathfrak{g}$-module in a natural way, with highest weight Λ. For the example of $\mathfrak{sl}_3$, the positive root generators are realized as

$$\begin{aligned}
e_{\alpha_1} &= \beta_{\alpha_1} \\
e_{\alpha_2} &= \beta_{\alpha_2} - \gamma^{\alpha_1} \beta_{\alpha_3} \\
e_{\alpha_3} &= \beta_{\alpha_3} \,,
\end{aligned} \qquad (3.2)$$

For $\Lambda \in P_+$ there exists a complex of such Fock modules

$$0 \longrightarrow \mathcal{F}_\Lambda^{(0)} \xrightarrow{\delta^{(0)}} \mathcal{F}_\Lambda^{(1)} \xrightarrow{\delta^{(1)}} \ldots \xrightarrow{\delta^{(s-1)}} \mathcal{F}_\Lambda^{(s)} \longrightarrow 0 \,,$$

where $s = |\Delta_+|$ and

$$\mathcal{F}_\Lambda^{(i)} = \bigoplus_{w \in W, \, \ell(w)=i} F_{w(\Lambda+\rho)-\rho} \,, \qquad (3.3)$$

which gives a resolution of the irreducible module $L(\Lambda)$. The differential of the complex is constructed from the so-called "screeners." For $\mathfrak{sl}_3$ the screeners are given by

$$
\begin{aligned}
s_{\alpha_1} &= -\beta_{\alpha_1} + \gamma^{\alpha_2} \beta_{\alpha_3} \,, \\
s_{\alpha_2} &= -\beta_{\alpha_2} \,, \\
s_{\alpha_3} &= -\beta_{\alpha_3} \,.
\end{aligned}
\qquad (3.4)
$$

Applying this resolution, we may proceed with the usual manipulations on the ensuing double complex $(\mathcal{F} \otimes F^{bc} \otimes F^{\sigma\omega}, d, \delta)$. The first spectral sequence associated with this double complex collapses at the second term,

$$
\begin{aligned}
E_\infty^{p,q} \cong E_2^{p,q} &\cong H^p(\mathfrak{n}_+, H^q(\delta, \mathcal{F}) \otimes \bigwedge \mathfrak{b}_-) \\
&\cong \delta^{q,0} \, H^p(\mathfrak{n}_+, L(\Lambda) \otimes \bigwedge \mathfrak{b}_-) \,,
\end{aligned}
\qquad (3.5)
$$

and produces the cohomology that we want to compute, while the E_2'-term for the second spectral sequence is given by

$$E_2'^{\,p,q} \cong H^q(\delta, H^p(\mathfrak{n}_+, \mathcal{F} \otimes \bigwedge \mathfrak{b}_-)) \,. \qquad (3.6)$$

Let us now restrict to the case of $\mathfrak{sl}_3$. To proceed it is convenient to first make a similarity transformation on d as follows: Introduce the operators

$$
\begin{aligned}
X &= -\gamma^{\alpha_1} c_1 b^{-\alpha_1} - \gamma^{\alpha_2} c_2 b^{-\alpha_2} - \gamma^{\alpha_3}(c_1 + c_2) b^{-\alpha_3} - \gamma^{\alpha_1} \gamma^{\alpha_2} c_2 b^{-\alpha_3} \,, \\
Y &= \gamma^{\alpha_1} c_{-\alpha_2} b^{-\alpha_3} - \gamma^{\alpha_2} c_{-\alpha_1} b^{-\alpha_3} \,,
\end{aligned}
\qquad (3.7)
$$

then

$$e^Y e^X \, d \, e^{-X} e^{-Y} = \sigma^\alpha e_\alpha - \sigma^{\alpha_1} \sigma^{\alpha_2} \omega_{\alpha_3} \,. \qquad (3.8)$$

This shows

$$E_1'^{\,p,q} \cong H^p(\mathfrak{n}_+, \mathcal{F} \otimes \bigwedge \mathfrak{b}_-) \cong H^p(\mathfrak{n}_+, \mathcal{F}^{(q)}) \otimes \bigwedge \mathfrak{b}_- \cong \delta^{p,0} \bigoplus_{w \in W \, \ell(w)=q} \left(\mathbb{C}_{w(\Lambda+\rho)-\rho} \otimes \bigwedge \mathfrak{b}_- \right) \,. \quad (3.9)$$

Having done this, at each Fock space in the original resolution we simply have a copy of $\bigwedge \mathfrak{b}_-$, and we must calculate the cohomology of the similarity-transformed δ on such. The operators $\delta^{(i)}$, in turn, are made up of similarity-transformed screeners. Since the screeners now operate on states with no γ's, we drop their β dependence in writing the transformed result below ($s_i = e^Y e^X s_{\alpha_i} e^{-X} e^{-Y}$).

$$
\begin{aligned}
s_1 &= c_1 b^{-\alpha_1} - c_{-\alpha_2} b^{-\alpha_3} \,, \\
s_2 &= c_2 b^{-\alpha_2} + c_{-\alpha_1} b^{-\alpha_3} \,, \\
s_3 &= (c_1 + c_2) b^{-\alpha_3} \,.
\end{aligned}
\qquad (3.10)
$$

Explicitly, the resolution is given by

$$
\begin{array}{ccc}
& F_{r_1} \longrightarrow F_{r_1 r_2} & \\
\nearrow & & \searrow \\
F_\Lambda & \times & F_{r_1 r_2 r_1} \\
\searrow & & \nearrow \\
& F_{r_2} \longrightarrow F_{r_2 r_1} &
\end{array}
$$

where $F_w \equiv F_{w(\Lambda+\rho)-\rho}$ and the intertwiners $Q_{w,w'} : F_w \to F_{w'}$ are given by

$$
\begin{aligned}
Q_{w,r_i w} &= s_i^{(\Lambda+\rho,w\alpha_i)}, \qquad \text{if } \ell(r_i w) = \ell(w) + 1\,, \\
Q_{r_1,r_1 r_2} &= \sum_{0 \le j \le l_2} b(l_2, l_1 + l_2; j)(s_2)^{l_2-j}(s_3)^j (s_1)^{l_2-j}\,, \\
Q_{r_2,r_2 r_1} &= \sum_{0 \le j \le l_1} b(l_1, l_1 + l_2; j)(s_1)^{l_1-j}(-s_3)^j (s_2)^{l_1-j}\,,
\end{aligned}
\tag{3.11}
$$

where $l_i = (\Lambda + \rho, \alpha_i)$ and

$$
b(m,n;j) = \frac{m!\,n!}{j!\,(m-j)!\,(n-j)!}\,.
\tag{3.12}
$$

It is now clear that since $s_i^n = 0$ for $i = 1,2$, $n \ge 3$, and $s_3^n = 0$ for $n \ge 2$ as well as $s_i^n s_3 = s_3 s_i^n = 0$ for $i = 1,2$, $n \ge 2$, all the differentials $\delta^{(i)}$ on E_1' vanish if $(\Lambda, \alpha_i) \ge 2$ for $i = 1,2$. Thus the spectral sequence collapses at E_1', and leads to a result consistent with that of Theorem 3.1. In the remaining cases the spectral sequence collapses at E_2' and the result is obtained by straightforward algebra. To formulate the answer in an elegant way let us introduce an extension $\widetilde{W}$ of the Weyl group W of $\mathfrak{sl}_3$ by $\widetilde{W} \equiv W \cup \{\sigma_1, \sigma_2\}$, and extend the length function on W by assigning $\ell(\sigma_1) = 1$ and $\ell(\sigma_2) = 2$. Let $\widetilde{W}$ act on $\mathfrak{h}^*$ by defining $\sigma_i \lambda = 0$, $i = 1,2$. We can now parametrize the weights in $\bigwedge \mathfrak{n}_-$ by $\sigma \in \widetilde{W}$ as follows

$$
P(\textstyle\bigwedge^n \mathfrak{n}_-) = \{\sigma\rho - \rho \,|\, \sigma \in \widetilde{W}, \ell(\sigma) = n\}\,.
\tag{3.13}
$$

We now have

Theorem 3.2. *For $\mathfrak{g} \cong \mathfrak{sl}_3$, the cohomology $H(\mathfrak{n}_+, L(\Lambda) \otimes \bigwedge \mathfrak{b}_-)_{\Lambda'}$ is nontrivial only if there exists a $w \in W$ and $\sigma \in \widetilde{W}$ such that $\Lambda' = (w(\Lambda + \rho) - \rho) + (\sigma\rho - \rho)$. The set of of allowed pairs (w, σ) depends on Λ and is given in the table below. For each allowed pair (w, σ) there is a quartet of cohomology states at ghost numbers $n, n+1, n+1$ and $n+2$ where $n = \ell(w) + \ell(\sigma)$.*

$w \backslash \sigma$	1	r_1	r_2	σ_1	$r_1 r_2$	$r_2 r_1$	σ_2	$r_1 r_2 r_1$
1	–	$m_1 \ge 1$	$m_2 \ge 1$	$m_1 \ge 1, m_2 \ge 1$	$m_1 \ge 2$	$m_2 \ge 2$	–	$m_1 \ge 1, m_2 \ge 1$
r_1	–	–	$m_1 \ge 1$	$m_2 \ge 1$	$m_2 \ge 1$	$m_2 \ge 1$	$m_1 \ge 1$	–
r_2	–	$m_2 \ge 1$	–	$m_1 \ge 1$	$m_1 \ge 1$	$m_1 \ge 1$	$m_2 \ge 1$	–
$r_1 r_2$	–	–	$m_1 \ge 1$	$m_1 \ge 1$	–	$m_1 \ge 1$	$m_2 \ge 1$	–
$r_2 r_1$	–	$m_2 \ge 1$	–	$m_2 \ge 1$	$m_2 \ge 1$	–	$m_1 \ge 1$	–
$r_1 r_2 r_1$	–	$m_1 \ge 2$	$m_2 \ge 2$	–	$m_1 \ge 1$	$m_2 \ge 1$	$m_1 \ge 1, m_2 \ge 1$	–

Table 3.1. Condition on Λ for the pair (w, σ) to be allowed
($m_i = (\Lambda, \alpha_i)$ and – means there's no condition on $\Lambda \in P_+$).

For Λ in the bulk, the quartet structure of $H(\mathfrak{n}_+, L(\Lambda) \otimes \bigwedge \mathfrak{b}_-)$ corresponds to the decomposition

$$
H(\mathfrak{n}_+, L(\Lambda) \otimes \textstyle\bigwedge \mathfrak{b}_-) \cong (H(\mathfrak{n}_+, L(\Lambda)) \otimes \textstyle\bigwedge \mathfrak{n}_-) \otimes \textstyle\bigwedge \mathfrak{h}\,.
\tag{3.14}
$$

It is quite remarkable that the quartet structure persists even away from the bulk because, e.g., it is not true in general that $H(\mathfrak{n}_+, L(\Lambda) \otimes \bigwedge \mathfrak{b}_-) \cong H(\mathfrak{n}_+, L(\Lambda) \otimes \bigwedge \mathfrak{n}_-) \otimes \bigwedge \mathfrak{h}$.

4. Comparison to $\mathcal{W}$-cohomology

Consider the $\mathcal{W}$-algebra $\mathcal{W}[\mathfrak{g}]$ associated to the simple, simply-laced, finite dimensional Lie algebra $\mathfrak{g}$ by means of the quantized Drinfel'd-Sokolov reduction with respect to the principal $\mathfrak{sl}_2$ embedding in $\mathfrak{g}$ (see, e.g., [16] and references therein). We will be mainly interested in the cases $\mathcal{W}_n$, $n = 2, 3$, where $\mathcal{W}_n \equiv \mathcal{W}[\mathfrak{sl}_n]$. The algebras $\mathcal{W}[\mathfrak{g}]$ have a realization in terms of ℓ free scalar fields coupled to a background charge α_0. The corresponding $\mathcal{W}[\mathfrak{g}]$-modules are the Fock spaces $F(\Lambda, \alpha_0)$. They are parametrized by the background charge α_0 (in terms of which the central charge is given by $c = \ell - 12\alpha_0{}^2|\rho|^2$) and a $\mathfrak{g}$-weight Λ.

Physical states of $\mathcal{W}[\mathfrak{g}]$ gravity coupled to ℓ free scalar fields (i.e. $\mathcal{W}[\mathfrak{g}]$ matter) are given by the semi-infinite cohomology of $\mathcal{W}[\mathfrak{g}]$ with coefficients in $F(\Lambda^M, \alpha_0^M) \otimes F(\Lambda^L, \alpha_0^L)$, i.e. $H(\mathcal{W}[\mathfrak{g}], F(\Lambda^M, \alpha_0^M) \otimes F(\Lambda^L, \alpha_0^L))$, where $F(\Lambda^M, \alpha_0^M)$ and $F(\Lambda^L, \alpha_0^L)$ represent the matter and gravity sector, respectively. A necessary condition for the semi-infinite cohomology to be defined is $(\alpha_0^M)^2 + (\alpha_0^L)^2 = -4$.

A particularly interesting case, commonly refered to as the noncritical $\mathcal{W}[\mathfrak{g}]$ string, is when the background charge in the matter sector vanishes (this corresponds to $c^M = \ell$). A subsector of this $\mathcal{W}[\mathfrak{g}]$ string, defined by restricting the momenta $(\Lambda^M, -i\Lambda^L)$ to lie on the lattice $L \equiv \{(\lambda, \mu) \in P \times P \,|\, \lambda - \mu \in Q\}$, possesses the structure of a BV-algebra. More precisely, let $\mathcal{C}$ be the complex

$$\mathcal{C} \equiv \bigoplus_{(\Lambda^M, -i\Lambda^L) \in L} F(\Lambda^M, \alpha_0^M) \otimes F(\Lambda^L, \alpha_0^L) \otimes F^{gh}, \tag{4.1}$$

where F^{gh} denotes the Fock space of the $\mathcal{W}[\mathfrak{g}]$ ghosts. Note that by summing the matter momenta over the integral weight lattice P, we have arranged that $\mathcal{C}$ becomes a $\widehat{\mathfrak{g}}$-module at level 1. In fact, the chiral algebra $\mathfrak{C}$ corresponding to $\mathcal{C}$ can be equipped with the structure of a Vertex Operator Algebra. The chiral algebra $H(\mathcal{W}[\mathfrak{g}], \mathfrak{C})$ corresponding to the semi-infinite $\mathcal{W}[\mathfrak{g}]$-cohomology of the complex $\mathcal{C}$ inherits the structure of a BV-algebra, where the product is given by the normal ordered product of operators and the BV-operator is given by $b_0^{[2]}$ – the zero mode of the anti-ghost corresponding to the Virasoro subalgebra of $\mathcal{W}[\mathfrak{g}]$. In addition, $H(\mathcal{W}[\mathfrak{g}], \mathfrak{C})$ is a $\mathfrak{g}^M \oplus \mathfrak{h}^L$ module, where the $\mathfrak{g}$ structure is a remnant of the $\widehat{\mathfrak{g}}$-module structure in the matter sector and $\mathfrak{h}$ is the action of the Liouville momenta.

The main result of this letter is the following

Conjecture 4.1. *We have an isomorphism of BV-algebras*

$$H(\mathcal{W}[\mathfrak{g}], \mathfrak{C}) \;\cong\; H(\mathfrak{n}_+, \mathcal{E}(G) \otimes \textstyle\bigwedge \mathfrak{b}_-) \;\equiv\; BV[\mathfrak{g}]. \tag{4.2}$$

We have checked that the conjecture is true for $\mathfrak{g} \cong \mathfrak{sl}_2$, i.e. for the Virasoro algebra, where the cohomology on the left hand side of (4.2) was computed in [8,9] and the BV-structure was unraveled in [3], by explicit comparison. For $\mathfrak{g} \cong \mathfrak{sl}_3$ both the cohomology and the BV-structure on the left hand side of (4.2) were determined in [6]. Again we find complete agreement with Conjecture 4.1. Note that the fact that $BV[\mathfrak{g}]$ is cyclic with respect to the BV-operator Δ (Conjecture 2.4) is crucial in making the identification $\Delta = b_0^{[2]}$.

To compare the cohomologies one has to remember that the action of $\mathfrak{h}^L$ on the left hand side of (4.2) is conventionally identified in the literature with the action of $-w_0(\Pi_i)$ on the right hand side. Also, to compare the $\mathfrak{sl}_3$ result with Table 3.2 in [6] one has to substitute $w \to w^{-1}$ and $\sigma \to w^{-1}\sigma w_0$.

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